Branchwidth of graphic matroids.

Frédéric Mazoit, Stéphan Thomassé

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Frédéric Mazoit* and Stéphan Thomassé†

Abstract

Answering a question of Geelen, Gerards, Robertson and Whittle [2], we prove that the branchwidth of a bridgeless graph is equal to the branchwidth of its cycle matroid. Our proof is based on branch-decompositions of hypergraphs.

1 Introduction.

Let $H = (V, E)$ be a hypergraph and (E_1, E_2) be a partition of E. The *border* of (E_1, E_2) is the set of vertices $\delta(E_1, E_2)$ which belong to both an edge of E_1 and an edge of E_2. We often write it $\delta(E_1, E_2)$, or often simply $\delta(E_1)$. A *component* of E is a minimum nonempty subset $C \subseteq E$ such that $\delta(C) = \emptyset$. Let F be a subset of E. We denote by $c(F)$ the number of components of the subhypergraph of H spanned by F, i.e. the hypergraph $(V(F), F)$. A hypergraph H is *2-edge connected* if every vertex belongs to at least two edges and $c(E \setminus e) = 1$ for every $e \in E$.

A *branch-decomposition* $\mathcal{T}$ of H is a ternary tree $\mathcal{T}$ and a bijection from the set of leaves of $\mathcal{T}$ into the set of edges of H. Practically, we simply identify the leaves of $\mathcal{T}$ to the edges of H. Observe that every edge e of $\mathcal{T}$ partitions $\mathcal{T} \setminus e$ into two subtrees, and thus correspond to a bipartition of E, called *e-separation*. More generally, a $\mathcal{T}$-separation is an e-separation for some edge e of $\mathcal{T}$. We will often identify the edge e of $\mathcal{T}$ with the e-separation, allowing us to write, for instance, $\delta(e)$ instead of $\delta(E_1, E_2)$, where (E_1, E_2) is the e-separation.

Let f be a real function defined on the set of bipartitions of E. For sake of simplicity we often write $f(E_1)$ instead of $f(E_1, E \setminus E_1)$. Let $\mathcal{T}$ be a branch-decomposition of H. The *f-width* of $\mathcal{T}$, denoted by $\mathrm{w}_f(\mathcal{T})$, is the maximum value of $f(e)$, for all edges e of T. The *f-branchwidth* of H, denoted by $\mathrm{bw}_f(H)$, is the minimum f-width of a branch-decomposition of H. A branch-decomposition achieving $\mathrm{bw}_f(H)$ is *f-optimal*.

The $|\delta|$-branchwidth (i.e. when $f(E_1, E_2) = |\delta(E_1, E_2)|$) of a graph G is the usual branchwidth introduced by Robertson and Seymour in [5]. In this paper, we study the branchwidth associated to the function $\rho(E_1, E_2) = |\delta(E_1, E_2)| +$

*LIF - CMI, 39 rue Joliot-Curie, 13453 Marseille Cedex 13, France. Frederic.Mazoit@lif.univ-mrs.fr

†Projet Mascotte, CNRS/INRIA/UNSA, INRIA Sophia-Antipolis, 2004 route des Lucioles BP 93, 06902 Sophia-Antipolis Cedex, France. thomasse@univ-lyon1.fr

$2 - c(E_1) - c(E_2)$. Our goal is to prove that in the class of 2-edge connected hypergraphs, the $|\delta|$-branchwidth is equal to the ρ-branchwidth. The proof simply consists to show that every 2-edge connected hypergraph admits a ρ-optimal decomposition such that $c(E_1) = c(E_2) = 1$ for every $\mathcal{T}$-separation (E_1, E_2).

Our motivation comes from the following: Let M be a matroid on base set E with rank function r. The *weight* of every non-trivial partition (E_1, E_2) of E is $w(E_1, E_2) := r(E_1)+r(E_2)-r(E)+1$. When $\mathcal{T}$ is a branch-decomposition of M, i.e. a ternary tree whose leaves are labelled by E, the *width* of $\mathcal{T}$ is the maximum weight of a $\mathcal{T}$-separation. Again, the *branchwidth* of M is the minimum width of a branch-decomposition of M. Let M be the cycle matroid of a 2-edge connected graph G, i.e. the matroid which base set is the set of edges of a graph and which independent sets are the acyclic subsets of edges. Branch-decompositions of G are exactly branch-decompositions of M. Moreover, $r(E_1) + r(E_2) - r(E) + 1$ is exactly $n_1 - c(E_1) + n_2 - c(E_2) - n + c(E) + 1$ where n_1, n_2, n are the number of vertices respectively spanned by E_1, E_2, E. Thus we have $w(E_1, E_2) = \delta(E_1) + 2 - c(E_1) - c(E_2) = \rho(E_1, E_2)$. In particular, the branchwidth of M is exactly the ρ-branchwidth, and thus is equal to the $|\delta|$-branchwidth.

Unless stated otherwise, we always assume that H is a 2-edge connected hypergraph and $\mathcal{T}$ is a branch-decomposition of H. Also, when speaking about width, branchwidth, etc, we implicitely mean ρ-width, ρ-branchwidth, etc.

2 Faithful branch-decompositions.

Let (E_1, E_2) be a $\mathcal{T}$-separation. The decomposition $\mathcal{T}$ is *faithful* to E_1 if for every component C of E_1, the partition $(C, E \setminus C)$ is a $\mathcal{T}$-separation. The *border graph* $G_{\mathcal{T}}$ has vertex set V and contains all edges xy for which there exists an edge e of $\mathcal{T}$ such that $\{x, y\} \subseteq \delta(e)$. A branch-decomposition $\mathcal{T}'$ is *tighter* than $\mathcal{T}$ if $\mathrm{w}_\rho(\mathcal{T}') < \mathrm{w}_\rho(\mathcal{T})$ or if $\mathrm{w}_\rho(\mathcal{T}) = \mathrm{w}_\rho(\mathcal{T}')$ and $G_{\mathcal{T}'}$ is a subgraph of $G_{\mathcal{T}}$. Moreover, $\mathcal{T}'$ is *strictly tighter* than $\mathcal{T}$ if $\mathcal{T}'$ is tighter than $\mathcal{T}$, and $\mathcal{T}$ is not tighter than $\mathcal{T}'$. Finally, $\mathcal{T}$ is *tight* if no $\mathcal{T}'$ is strictly tighter than $\mathcal{T}$.

Lemma 1 *Let (E_1, E_2) be a partition of E. For any union E'_1 of connected components of E_1 and E_2, we have both $\delta(E'_1) \subseteq \delta(E_1)$ and $\rho(E'_1) \leq \rho(E_1)$.*

$\square$ Clearly, $\delta(E'_1) \subseteq \delta(E_1)$. Moreover, every vertex of $\delta(E_1)$ belongs to one component of E_1 and one component of E_2. Therefore, if C is a component of E'_1 which is the union of k components of E_1 and E_2, there are at least $k - 1$ vertices of $C \setminus \delta(C)$ which belong to $\delta(E_1)$, In all, the weight of the separation increased by $k - 1$ since we merge k components into one, but it also decreased by at least $k - 1$ since we lose at least that many vertices on the border. Since this is the case for every component of E'_1 or of $E \setminus E'_1$, we have $\rho(E'_1) \leq \rho(E_1)$.

■

Lemma 2 *Let (E_1, E_2) be an e-separation of $\mathcal{T}$. Let $\mathcal{T}_1$ be the subtree of $\mathcal{T} \setminus e$ with set of leaves E_1. If $\mathcal{T}$ is not faithful to E_1, one can modify $\mathcal{T}_1$ in $\mathcal{T}$ to form a tighter branch-decomposition $\mathcal{T}'$ of H.*

$\square$ Fix the vertex $e \cap \mathcal{T}_1$ as a root of $\mathcal{T}_1$. Our goal is to change the binary rooted tree $\mathcal{T}_1$ into another binary rooted tree $\mathcal{T}'_1$. For every connected component C of E_1, consider the subtree $\mathcal{T}_C$ of $\mathcal{T}_1$ which contains the root of $\mathcal{T}_1$ and has set of leaves C. Observe that $\mathcal{T}_C$ is not necessarily binary since $\mathcal{T}_C$ may contain paths having internal vertices with only one descendant. We simply replace these paths by edges to obtain our rooted tree $\mathcal{T}'_C$. Now, consider any rooted binary tree BT with $c(E_1)$ leaves and identify these leaves to the roots of $\mathcal{T}'_C$, for all components C of E_1. This rooted binary tree is our $\mathcal{T}'_1$. We denote by $\mathcal{T}'$ the branch-decomposition we obtain from $\mathcal{T}$ by replacing $\mathcal{T}_1$ by $\mathcal{T}'_1$. Roughly speaking, we merged all subtrees of $\mathcal{T}_1$ induced by the components of E_1 together with $\mathcal{T} \setminus \mathcal{T}_1$ to form $\mathcal{T}'$. Let us prove that $\mathcal{T}'$ is tighter than $\mathcal{T}$. For this, consider an edge f' of $\mathcal{T}'$. If $f' \notin \mathcal{T}'_1$, the f'-separations of $\mathcal{T}$ and $\mathcal{T}'$ are the same. If $f' \in BT$, by Lemma 1, we have $\rho(f') \leq \rho(e)$ and $\delta(f') \subseteq \delta(e)$. So the only case we have to care of is when f' is an edge of some tree $\mathcal{T}'_C$, where C is a component of E_1. Recall that f' corresponds to a path P of $\mathcal{T}_C$. Let f be any edge of P. Let $(F, E \setminus F)$ be the f-separation of $\mathcal{T}$, where $F \subseteq E_1$. Therefore, the f'-separation of $\mathcal{T}'$ is $\big(F \cap C, E \setminus (F \cap C)\big)$. Since F is a subset of E_1, the connected components of F are subsets of the connected components of E_1. Thus $F \cap C$ is a union of connected components of F. This implies that $\delta(f') \subseteq \delta(f)$. Also, by Lemma 1, $\rho(f') \leq \rho(f)$.

We have proved that $\mathrm{w}(\mathcal{T}') \leq \mathrm{w}(\mathcal{T})$ and that $G_{\mathcal{T}'}$ is a subgraph of $G_{\mathcal{T}}$, thus $\mathcal{T}'$ is tighter than $\mathcal{T}$. $\blacksquare$

3 Connected branch-decompositions.

Let $F \subseteq E$ be a component. The hypergraph $H * F$ on vertex set V and edge set $(E \setminus F) \cup \{V(F)\}$ is denoted by $H * F$. In other words, $H * F$ is obtained by merging the edges of F into one edge. A partition (E_1, E_2) of E is *connected* if $c(E_1) = c(E_2) = 1$. A branch-decomposition $\mathcal{T}$ is *connected* if all its $\mathcal{T}$-separations are connected.

Lemma 3 *If $\mathcal{T}$ is tight, every $\mathcal{T}$-separation (E_1, E_2) is such that E_1 or E_2 is connected.*

$\square$ Suppose for contradiction that there exists a $\mathcal{T}$-separation (E_1, E_2) such that neither E_1 nor E_2 is connected. By Lemma 2, we can assume that $\mathcal{T}$ is faithful to E_1 and to E_2. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be respectively the sets of components of E_1 and E_2. Consider the graph on set of vertices $\mathcal{C}_1 \cup \mathcal{C}_2$ where C_1C_2 is an edge whenever $C_1 \in \mathcal{C}_1$ and $C_2 \in \mathcal{C}_2$ have nonempty intersection. This graph is connected and is not a star. Thus, it has a vertex-partition into two connected subgraphs, each having at least two vertices. This vertex-partition corresponds to a partition $(\mathcal{C}'_1, \mathcal{C}'_2)$ of $\mathcal{C}_1 \cup \mathcal{C}_2$.

Consider any rooted binary tree BT with $|\mathcal{C}'_1|$ leaves. Since every $C \in \mathcal{C}'_1$ is an element of $\mathcal{C}_1 \cup \mathcal{C}_2$, $(C, E \setminus C)$ is an e-separation of $\mathcal{T}$. We denote by $\mathcal{T}_C$ the tree of $\mathcal{T} \setminus e$ with set of leaves C. Root $\mathcal{T}_C$ with the vertex $e \cap \mathcal{T}_C$ in order to get a binary rooted tree. Now identify the leaves of BT with the roots of $\mathcal{T}_C$, for $C \in \mathcal{C}'_1$. This rooted tree is our $\mathcal{T}'_1$. We construct similarly $\mathcal{T}'_2$. Adding an edge between the roots of $\mathcal{T}'_1$ and $\mathcal{T}'_2$ gives the branch-decomposition $\mathcal{T}'$ of H. By Lemma 1, $\mathrm{w}(\mathcal{T}') \leq \mathrm{w}(\mathcal{T}')$. Moreover, $G_{\mathcal{T}'}$ is a subgraph of $G_{\mathcal{T}}$. Let us now show that $G_{\mathcal{T}'}$ is a strict subgraph of $G_{\mathcal{T}}$. Indeed, since $\mathcal{C}'_1$ is connected and has at least two elements, it contains $C_1 \in \mathcal{C}_1$ and $C_2 \in \mathcal{C}_2$ such that $C_1 \cap C_2$ is nonempty. By construction, every vertex x of $C_1 \cap C_2$ is such that $x \notin \delta(\mathcal{C}'_1)$ and $x \in \delta(\mathcal{C}_1)$. Similarly, there is a vertex y spanned by $\mathcal{C}'_2$ such that $y \notin \delta(\mathcal{C}'_2)$ and $y \in \delta(\mathcal{C}_2)$. Thus xy is an edge of $G_{\mathcal{T}}$ but not of $G_{\mathcal{T}'}$, contradicting the fact that $\mathcal{T}$ is tight. ■

Theorem 1 *For every branch-decomposition $\mathcal{T}$ of a hypergraph H, there exists a tighter branch-decomposition $\mathcal{T}'$ such that for every $\mathcal{T}'$-separation (E_1, E_2) with $c(E_1) > 1$, E_1 consists of components of $H \setminus e$, for some $e \in E$. In particular, if H is 2-edge connected, it has an optimal connected branch-decomposition.*

□ Let us prove the theorem by induction on $|V|+|E|$. The statement is obvious if $|E| \leq 3$, so we assume now that H has at least four edges. Call *achieved* a branch-decomposition satisfying the conclusion of Theorem 1. If $\mathcal{T}$ is not tight, we can replace it by a tight branch-decomposition tighter than $\mathcal{T}$. So we may assume that $\mathcal{T}$ is tight.

If H is not connected, apply induction on every components of H in order to find an achieved branch-decomposition. Then merge these branch-decompositions into one branch-decomposition of H.

If there is an edge $e \in E$ such that $H \setminus e$ is not connected, we can assume by Lemma 2 that $\mathcal{T}$ is faithful to $E \setminus e$. Let E_1 be a connected component of $E \setminus e$. Let $\mathcal{T}_1$ be the branch-decomposition induced by $\mathcal{T}$ on $E_1 \cup e$. Let also $\mathcal{T}_2$ be the branch-decomposition induced by $\mathcal{T}$ on $(E \setminus E_1) \cup e$. By the induction hypothesis, there exists two achieved branch-decompositions $\mathcal{T}'_1$ and $\mathcal{T}'_2$, respectively tighter than $\mathcal{T}_1$ and $\mathcal{T}_2$. Identify the leaf e of the trees $\mathcal{T}'_1$ and $\mathcal{T}'_2$, and attach a leaf labelled by e to the identified vertex. Call $\mathcal{T}'$ this branch-decomposition of H. Observe that it is tighter than $\mathcal{T}$ and achieved.

So we assume now that H is 2-edge connected. The key-observation is that if there is a connected $\mathcal{T}$-separation (E_1, E_2) with $|E_1| \geq 2$ and $|E_2| \geq 2$, we can apply the induction hypothesis on $H * E_1$ and $H * E_2$ and merge the two branch-decompositions to obtain an optimal connected branch-decomposition of H. Therefore, we assume that every $\mathcal{T}$-separation (E_1, E_2) with $|E_1| \geq 2$ and $|E_2| \geq 2$ is such that E_1 or E_2 is connected.

We now orient the edges of $\mathcal{T}$. If (E_1, E_2) is an e-separation such that E_2 is connected but not an edge of H, we orient e from E_1 to E_2. Since H is 2-edge-connected, every edge of $\mathcal{T}$ incident to a leaf is oriented from the leaf. By Lemma 3, every edge get at least one orientation. And by the key-observation, every edge of $\mathcal{T}$ has exactly one orientation.

This orientation of $\mathcal{T}$ has no circuit, thus there is a vertex $t \in T$ with outdegree zero. Since every leaf has outdegree one, t has indegree three. Let us denote by A, B, C the set of leaves of the three trees of $\mathcal{T} \setminus t$. Observe that by construction, $A \cup B$, $A \cup C$ and $B \cup C$ are connected. By Lemma 2, we can assume moreover that $\mathcal{T}$ is faithful to A, B and C. We claim that A is a disjoint union of edges, i.e. the connected components of A are edges of H. To see this, pick any component C_A of A. Since $\mathcal{T}$ is faithful to A, $(C_A, E \setminus C_A)$ is a $\mathcal{T}$-separation. But this is simply impossible since $B \cup C$ being connected, $E \setminus C_A$ is also connected, against the fact that every edge has a single orientation. So the hypergraph H consists of three sets of disjoint edges A, B, C. Call this partition the *canonical partition* of $\mathcal{T}$. Call $(A, E \setminus A)$, $(B, E \setminus B)$ and $(C, E \setminus C)$ the *main* $\mathcal{T}$-separations. The width of every other $\mathcal{T}$-separation is strictly less than $\mathrm{bw}(H)$. Since every vertex of H belongs to two or three edges, it is spanned by at least two of the sets $\delta(A), \delta(B), \delta(C)$. In particular $G_{\mathcal{T}}$ is a complete graph, and thus every optimal branch-decomposition of H is tighter than $\mathcal{T}$. Set $\delta_{AB} := |\delta(A) \cap \delta(B)|$, $\delta_{AC} := |\delta(A) \cap \delta(C)|$, $\delta_{BC} := |\delta(B) \cap \delta(C)|$ and $\delta_{ABC} := |\delta(A) \cap \delta(B) \cap \delta(C)|$. We now prove some properties of H.

1. Two of the sets A, B, C have at least two edges. Indeed, assume for contradiction that $A = \{a\}$ and $B = \{b\}$. Since $|E| \geq 4$, there are at least two edges in C. Let $c \in C$. Assume without loss of generality that $|a \cap c| \geq |b \cap c|$. Now form a new branch-decomposition $\mathcal{T}'$ by *moving* c to A, i.e. $\mathcal{T}'$ has a separation $(A \cup c, B \cup (C \setminus c))$ and then four branches $A, c, B, (C \setminus c)$. We have

$$\rho(A \cup c, B \cup (C \setminus c)) = \rho(A) + |b \cap c| - |a \cap c|$$

   since both parts are connected. In particular $\mathcal{T}'$ is tighter than $\mathcal{T}$, and since the $\mathcal{T}'$-separation $(A \cup c, B \cup (C \setminus c))$ is connected and both of its branches have at least two vertices, we can apply induction to conclude.

2. Every set A, B, C have at least two edges. Indeed, assume for contradiction that A consists of a single edge a. Let b be an edge of B. If $|b \cap \delta(C)| \leq |b \cap a|$, we can as previously move b to A in order to conclude. Call $|b \cap \delta(C)| - |b \cap a|$ the *excess* of b. Similarly, call $|c \cap \delta(B)| - |c \cap a|$ the *excess* of an edge $c \in C$. Let s be the minimum excess of an edge e_s of $B \cup C$. Observe that $s \geq 1$ and that every $b \in B$ satisfies $|b \cap \delta(C)| \geq |b \cap \delta(A)| + s$. Thus, summing for all edges of B, we obtain $\delta_{BC} \geq \delta_{AB} + s|B|$. Similarly, $\delta_{BC} \geq \delta_{AC} + s|C|$. Note also that $\mathrm{bw}(H) \geq \rho(C) = \delta_{BC} + \delta_{AC} - \delta_{ABC} - |C| + 1$ and $\mathrm{bw}(H) \geq \rho(B) = \delta_{BC} + \delta_{AB} - \delta_{ABC} - |B| + 1$. In all

$$2\,\mathrm{bw}(H) \geq 2\delta_{BC} - 2\delta_{ABC} + \delta_{AC} - |C| + \delta_{AB} - |B| + 2.$$

   Then $2\,\mathrm{bw}(H) \geq \delta_{AB} + s|B| + \delta_{AC} + s|C| - 2\delta_{ABC} + \delta_{AC} - |C| + \delta_{AB} - |B| + 2$. Finally, $\mathrm{bw}(H) \geq \delta_{AC} + \delta_{AB} - \delta_{ABC} + 1 + ((s-1)|C| + (s-1)|B|)/2$. Since $\rho(A) = \delta_{AC} + \delta_{AB} - \delta_{ABC}$, we have $\mathrm{bw}(H) \geq \rho(A) + s$. But then we can move e_s to A to conclude.

3. We have $\text{bw}(H) = \rho(A)$. If not, pick two edges a, a' of A and merge them together. The hypergraph we obtain is still 2-edge-connected, and the branch-decomposition still has the same width. Apply induction to get an achieved branch-decomposition. Then replace the merged edge by the two original edges. This branch-decomposition $\mathcal{T}'$ is optimal, hence tighter than $\mathcal{T}$. So either we can apply induction on $\mathcal{T}'$, or $\mathcal{T}'$ has a canonical partition. But in this last case, the canonical partition of $\mathcal{T}'$ is exactly $\{a\}$, $\{a'\}$, $E \setminus \{a, a'\}$. Thus by 1, we can apply induction. Similarly, $\text{bw}(H) = \rho(B) = \rho(C)$.

4. We have $\text{bw}(H) \geq \beta + 1$, where β is the size of a maximum edge e of H. Since edges of H with only one vertex play no role here, we can ignore them. So the size of an edge of H is at least two. Assume for instance that $e \in A$. Since A has at least two components, we have $\text{bw}(H) = \rho(A) \geq |\delta(A)| - c(B \cup C) - c(A) + 2 \geq 2 + \beta - 1$.

5. We have $\delta_{ABC} = 0$. Indeed, suppose for contradiction that there exists a vertex z in $\delta(A) \cap \delta(B) \cap \delta(C)$. Consider the hypergraph H_z obtained from H by removing the vertex z from all its edges. The branch-decomposition $\mathcal{T}$ induces a branch-decomposition $\mathcal{T}_z$ of H_z having width at most $\text{bw}(\mathcal{T}) - 1$. Observe also that H_z is connected since z is incident to three edges and H is 2-edge connected. We apply induction on $\mathcal{T}_z$ to obtain an achieved branch-decomposition $\mathcal{T}'_z$ of H_z. Now add back the vertex z to the edges of H_z and call $\mathcal{T}'$ the branch-decomposition obtained from $\mathcal{T}'_z$. Observe that if a $\mathcal{T}'_z$-separation (E_1, E_2) is connected, adding z will raise by at most one its width in $\mathcal{T}'$. Moreover if a $\mathcal{T}'_z$-separation (E_1, E_2) is not connected, say $c(E_2) > 1$, adding z can raise by at most two its width in $\mathcal{T}'$ (either by merging three components of E_2 into one, or by merging two and increasing the border by one). Since $\mathcal{T}'_z$ is achieved, E_2 is a set of components of $E \setminus e$ for some edge e of E. But then, $\rho_{\mathcal{T}'_z}(E_1, E_2) \leq |e| - 3 + 2 \leq \beta - 1 \leq \text{bw}(H) - 2$, and thus $\rho_{\mathcal{T}'}(E_1, E_2) \leq \text{bw}(H)$. Finally $\mathcal{T}'$ is optimal. Moreover every $\mathcal{T}'$-separation (E_1, E_2) is connected. Indeed, if E_1 is connected in $\mathcal{T}'_z$, we are done. If E_1 is not connected in $\mathcal{T}'_z$, E_1 consists of components of $H_z \setminus e$, for some edge e of H_z. But since H is 2-edge connected, every component of E_1 in H contains z, otherwise they would be components of $H \setminus e$. Consequently E_1 is connected.

6. Every edge of H is incident to at least four other edges. Indeed, assume for contradiction that an edge a of A is incident to only one edge b of B and at most two edges of C. Moving a to B increases $\rho(B)$ by $|a \cap \delta(C)| - |a \cap b|$ and does not increase $\rho(A)$ and $\rho(C)$. Therefore, if $|a \cap \delta(C)| \leq |a \cap b|$, we can move a to B, and this new branch-decomposition $\mathcal{T}'$ is strictly tighter than $\mathcal{T}$ since the vertices of $a \cap b$ are no more joined to $(\delta(A) \setminus a) \cap \delta(C)$ in the graph $G_{\mathcal{T}'}$. Thus $|a \cap \delta(C)| \geq |a \cap b| + 1$. Moreover, moving a to C, increases $\rho(C)$ by at most $|a \cap b| - |a \cap \delta(C)| + 1$, since at most two components of C can merge. So $|a \cap b| + 1 > |a \cap \delta(C)|$, a contradiction.

This implies in particular that $\rho(e) \leq \mathrm{bw}(H) - 3$ whenever e is not one of the main $\mathcal{T}$-separations. In particular $\beta \leq \mathrm{bw}(H) - 3$.

7. The hypergraph H is triangle-free. Indeed, suppose that there exists three edges $a \in A$, $b \in B$ and $c \in C$ and three vertices $x \in a \cap b$, $y \in b \cap c$ and $z \in c \cap a$. Let H_{xyz} be the hypergraph obtained by removing x, y, z in the vertex set of H and in every edge of H. The branchwidth of H_{xyz} is at most $\mathrm{bw}(H) - 2$, since we removed two vertices in the border of every main separation. As in 5, $\mathcal{T}$ induces a branch-decomposition $\mathcal{T}_{xyz}$ of H_{xyz} which can be improved by induction to an achieved branch-decomposition $\mathcal{T}'_{xyz}$ of H'_{xyz}. Adding back the vertices x, y, z, we obtain a branch-decomposition $\mathcal{T}'$ of H. We claim that $\mathrm{w}(\mathcal{T}') \leq \mathrm{bw}(H)$. Let (E_1, E_2) be a $\mathcal{T}'$-separation, without loss of generality, we assume that E_1 contains at least two edges of a, b, c, say a and b. Assume first that (E_1, E_2) is connected in $\mathcal{T}'_{xyz}$. If $c \in E_1$, we have $\rho_{\mathcal{T}'_{xyz}}(E_1) = \rho_{\mathcal{T}'}(E_1)$. If $c \notin E_1$, $\rho_{\mathcal{T}'_{xyz}}(E_1) + 2 = \rho_{\mathcal{T}'}(E_1)$ since we add the endvertices of c to the border of E_1. Now, if (E_1, E_2) is not connected in $\mathcal{T}'_{xyz}$, then $\rho_{\mathcal{T}'_{xyz}}(E_1)$ is at most the size of an edge of H_{xyz}, hence at most β. Since x, y, z have degree two in H, each can either increase the border of a separation by one, or merge two components. In all, $\rho_{\mathcal{T}'_{xyz}}(E_1)$ increases by at most three. Since $\beta \leq \mathrm{bw}(H) - 3$, we have that $\mathrm{w}(\mathcal{T}') \leq \mathrm{bw}(H)$. To conclude, we prove that $\mathcal{T}'$ is connected. Indeed, if $c(E_2) > 1$ for some $\mathcal{T}'$-separation (E_1, E_2), E_2 consists of components of $H_{xyz} \setminus e$ for some edge e of H_{xyz}. Since H is 2-edge connected, these components are not components of $H \setminus e$, so each of them must contain one of the edges a, b or c, and therefore E_2 is connected in $\mathcal{T}'$.

Now we are ready to finish the proof. Note that $\mathrm{bw}(H) = (\rho(A) + \rho(B) + \rho(C))/3 = (2|V| - |E|)/3 + 1$. Consider the line multigraph $L(H)$ of H, i.e. the multigraph on vertex set $A \cup B \cup C$ and edge set V such that $v \in V$ is the edge which joins the two edges e, f of H such that $v \in e$ and $v \in f$. The multigraph $L(H)$ satisfies the hypothesis of Lemma 4 (proved in the next section), thus it admits a vertex-partition of its vertices as in the conclusion of Lemma 4. This corresponds to a partition of $A \cup B \cup C$ into two subsets $E_1 := A_1 \cup B_1 \cup C_1$ and $E_2 := A_2 \cup B_2 \cup C_2$ such that $|\delta(E_1, E_2)| \leq (2|V| - |E|/3) + 1$ and both E_1 and E_2 have at least $\lfloor |E|/2 \rfloor - 1$ internal vertices. In particular, the separation (E_1, E_2) has width at most $\mathrm{bw}(H)$. Let us show that one of $\rho(A_1 \cup B_1)$, $\rho(B_1 \cup C_1)$, and $\rho(C_1 \cup A_1)$ is also at most $\mathrm{bw}(H)$. For this, observe that

$$\delta(A_1 \cup B_1) + \delta(B_1 \cup C_1) + \delta(C_1 \cup A_1) \leq 2(|V| - (|E_2| - |\delta(E_2)|)).$$

Thus $\delta(A_1 \cup B_1) + \delta(B_1 \cup C_1) + \delta(C_1 \cup A_1) \leq 2|V| - 2\lfloor |E|/2 \rfloor + 2 \leq 2|V| - |E| + 3$. Without loss of generality, we can assume that $\delta(A_1 \cup B_1) \leq (2|V| - |E|)/3 + 1 = \mathrm{bw}(H)$, and thus we split E_1 into two branches $A_1 \cup B_1$ and C_1. We similarly split E_2 to obtain an optimal branch-decomposition $\mathcal{T}'$ of H. Observe that since both $E_1 \setminus \delta(E_1)$ and $E_2 \setminus \delta(E_2)$ are not empty, the graph $G_{\mathcal{T}'}$ is not complete, against the fact that $\mathcal{T}$ is tight, a contradiction. ■

4 The technical Lemma.

Let G be a multigraph and X, Y two subsets of its vertices. We denote by $e(X, Y)$ the number of edges of G between X and Y. We also denote by $e(X)$ the number of edges in X.

Lemma 4 *Let G be a 2-connected triangle-free multigraph on $n \geq 5$ vertices and m edges. Assume that its minimum underlying degree (forgotting the multiplicity of edges) is four. Assume moreover that its maximum degree is at most $(2m - n)/3+1$. There exists a partition (X, Y) of the vertex set of G such that $e(X) \geq \lfloor n/2 \rfloor - 1$, $e(Y) \geq \lfloor n/2 \rfloor - 1$ and $e(X, Y) \leq (2m - n)/3 + 1$.*

$\Box$ Call *good* a partition which satisfies the conclusion of Lemma 4. Assume first that there are vertices x, y such that $e(x, y) \geq \lfloor n/2 \rfloor - 1$. The minimum degree in $V \setminus \{x, y\}$ is at least two, so $e(V \setminus \{x, y\})$ is at least $\lfloor n/2 \rfloor - 1$. Thus, if the partition $(V \setminus \{x, y\}, \{x, y\})$ is not good, we necessarily have $d(x) + d(y) - 2e(x, y) > (2m - n)/3 + 1$. By the maximum degree hypothesis, both $d(x)$ and $d(y)$ are greater than $2e(x, y)$. Since G is triangle-free, there exists a partition (X, Y) where $(N(x) \cup x) \setminus y \subseteq X$ and $(N(y) \cup y) \setminus x \subseteq Y$. Observe that $e(X) \geq d(x) - e(x, y) > e(x, y) \geq \lfloor n/2 \rfloor - 1$. Similarly $e(Y) \geq \lfloor n/2 \rfloor - 1$. Moreover, since $m \geq 2n$ by the minimum degree four hypothesis, we have

$$\begin{aligned} e(X, Y) &\leq m - (d(x) + d(y) - 2e(x, y)) \\ &< m - (2m - n)/3 - 1 \\ &\leq (m + n)/3 - 1 \\ &\leq (2m - n)/3 + 1. \end{aligned}$$

So (X, Y) is a good partition. We assume from now on that the multiplicity of an edge is less than $\lfloor n/2 \rfloor - 1$.

Let $a + b = n$, where $a \leq b$. A partition (X, Y) of V is an *a-partition* if $|X| \leq a$, $e(X) \geq a - 1$, $e(Y) \geq b - 1$, $e(X, Y) \leq (2m - n)/3 + 1$, and the additional requirement that X contains a vertex of G with maximum degree.

Note that there exists a 1-partition, just consider for this $X := \{x\}$, where x has maximum degree in G (the minimum degree in Y is at least three, insuring that $e(Y) \geq n - 2$). We consider now an a-partition (X, Y) with maximum a. If $a \geq b - 1$, this partition is good and we are done. So we assume that $a < b - 1$. In particular $e(X) = a - 1$. The *excess* of a vertex $y \in Y$ is $exc(y) := d_Y(y) - d_X(y)$.

The key-observation is that there exists at most one vertex $y \in Y$ such that $e(Y \setminus y) < b - 2$. Indeed, if there is a vertex of Y with degree one in Y, we simply move it to X, and we obtain an $(a + 1)$-partition ($e(X)$ increases, $e(Y)$ decreases by one, and $e(X, Y)$ decreases). Thus the minimum degree in Y is at least two. Moreover, if there is a vertex of Y with degree two, we can still move it to X ($e(X)$ increases, $e(Y) \geq |Y| - 2$ and $e(X, Y)$ does not increase). So the minimum degree in Y is at least three (but the underlying minimum degree may be one). This implies that $e(Y) \geq 3|Y|/2$. Let $Y := \{y_1, \ldots, y_{|Y|}\}$ where the vertices are indexed in the increasing order according to their degree in Y. For every $i \neq |Y|$, we have $e(Y) \geq (3(|Y|-2)+d_Y(y_i)+d_Y(y_{|Y|}))/2$. Furthermore,

$$\begin{aligned} e(Y \setminus y_i) &\geq (3(|Y|-2)+d_Y(y_i)+d_Y(y_{|Y|}))/2 - d_Y(y_i) \\ &\geq 3(|Y|-2)/2 \\ &\geq |Y|-2 \\ &\geq b-2. \end{aligned}$$

We now discuss the two different cases.

- Assume that $e(Y \setminus y) \geq b-2$ for every $y \in Y$. We denote by Y' the (nonempty) set of vertices of Y with at least one neighbour in X. Set $Y'' := Y \setminus Y'$. Denote by c the minimum excess of a vertex of Y'. Observe that every vertex of Y'' has degree at least four in Y. Thus summing the degrees of the vertices of Y gives

  $$2e(Y) \geq e(X,Y) + 4|Y''| + c|Y'| \qquad (1)$$

  Let $y \in Y'$ such that $exc(y) = c$. Since the partition $(X \cup y, Y \setminus y)$ is not an $(a+1)$-partition, we have $e(X,Y)+c > (2m-n)/3+1$. Since $m = e(X,Y)+e(X)+e(Y)$, this implies

  $$e(X,Y) + 3c > 2e(X) + 2e(Y) - n + 3 \qquad (2)$$

  Equations (1) and (2) give:

  $$e(X,Y) + 3c > 2e(X) + e(X,Y) + 4|Y''| + c|Y'| - n + 3 \qquad (3)$$

  Since $e(X) \geq n-|Y|-1$, we get $3c > n-2|Y|+4|Y''|+c|Y'|+1$. So $3c > n+2|Y|+(c-4)|Y'|+1$, and finally $n+2|Y| < (c-4)(3-|Y'|)+11$. If $c=4$, we get $n+2|Y| \leq 10$, impossible. If $c=3$, we get $n+2|Y|-|Y'| \leq 7$, impossible. If $c=2$, we get $n+2|Y|-2|Y'| \leq 4$, impossible. If $c=1$, we get $n+2|Y|-3|Y'| \leq 1$, which can only hold if $|Y|=|Y'|=n-1$. Thus, X consists of a single vertex, completely joined to Y, against the fact that G is triangle-free. Finally $c > 4$, and consequently $|Y'| < 3$. Observe that $|Y'| > 1$ since G is 2-connected. Thus $|Y'| = 2$. Let y_1, y_2 be the vertices of Y', indexed in such a way that $e(y_1,X)+e(y_2,Y'') \geq e(y_2,X)+e(y_1,Y'')$. Let $X_1 := X \cup y_1$ and $Y_1 := Y \setminus y_1$. Since $y_1 \in Y'$, we have that $e(X_1) \geq a$. Moreover $e(Y_1) \geq b-2$. Observe also that $e(y_1,y_2) \leq e(Y \setminus \{y_1,y_2\})$: this is obvious if $e(y_1,y_2)=0$, and if there is an edge between y_1 and y_2, since G is triangle-free with minimum degree four, the minimum degree in $Y \setminus \{y_1,y_2\}$ is at least two. So

  $$e(Y \setminus \{y_1,y_2\}) \geq |Y|-2 \geq \lfloor n/2 \rfloor - 1 \geq e(y_1,y_2).$$

  In all $e(X_1,Y_1) \leq e(X_1)+e(Y_1)$. And since $n-2 \leq e(X_1)+e(Y_1)$, we have $e(X_1,Y_1) \leq 2e(X_1)+2e(Y_1)-n+2$, which implies $e(X_1,Y_1) \leq (3m-n)/2+1$. So the partition (X_1,Y_1) is good.

- Now assume that there exists a vertex $y \in Y$ such that $e(Y \setminus y) \leq b-3$. We denote by Y' the set of vertices of $Y \setminus y$ with at least one neighbour in X. Set $Y'' := Y \setminus (Y' \cup y)$. Observe that since every vertex of Y'' has underlying degree four in Y, we have $e(Y \setminus y) \geq 3|Y''|/2$. Thus, $|Y''| \leq (2|Y|-6)/3$. Since $|Y| \geq 3$, we have $|Y''| \leq |Y|-3$, and finally $|Y'| \geq 3$. Denote by c the minimum excess of a vertex of Y'. Summing the degrees of the vertices of Y gives $2e(Y) \geq e(X,Y)+4|Y''|+c|Y'|+exc(y)$. Equation (2) still holds, so

  $$exc(y) < 3c+n-3-2e(X)-4|Y''|-c|Y'| \leq 3c-1-e(X)-3|Y''|-(c-1)|Y'|$$

  since $e(X)+|Y''|+|Y'| \geq n-2$. Therefore $exc(y) < -e(X)-3|Y''|-(c-1)(|Y'|-3)+2$. Since $|Y'| \geq 3$ and $c \geq 1$, we have $exc(y) \leq 1-e(X)$. Moreover, since the minimum degree in Y is at least three, summing the degrees in Y of the vertices of $Y \setminus y$ gives that $3(|Y|-1) \leq 2e(Y \setminus y) + d_Y(y) \leq 2b-6+d_Y(y)$. Finally, $d_Y(y) \geq |Y|+3$ and by the fact that $exc(y) \leq 1-e(X)$, we have $d_X(y) \geq |Y|+e(X)+2$. Recall that X contains a vertex x with maximum degree in G. In particular both x and y have degree at least $2|Y|+e(X)+5$. Observe that $d_Y(x)$ is at least $2|Y|+5$. Now the end of the proof is straightforward, it suffices to switch x and y to obtain the good partition $(X_1,Y_1) := ((X \cup y) \setminus x, (Y \cup x) \setminus y)$. The only fact to care of is $e(x,y)$. Indeed if $e(x,y)$ is at most $e(X)$, we have:

  1. $e(Y_1) \geq d_{Y_1}(x) \geq 2|Y|+5-e(x,y) \geq 2|Y|-e(X) \geq |Y| \geq n/2$.
  2. $e(X_1) \geq d_{X_1}(y) \geq |Y|+e(X)+2-e(x,y) \geq n/2$.
  3. Finally, since the excess of y is at most $1-e(X)$, we have $d_X(y) \geq d_Y(y)+e(X)-1$, hence $d_{X_1}(y) \geq d_Y(y)-1$. Moreover $d_{Y_1}(x) \geq 2|Y|+5-e(X) \geq e(X)+5 \geq d_X(x)+5$. In all, we have $e(X_1,Y_1) \leq e(X,Y)-4 \leq (2m-n)/3+1$, since (X,Y) is an a-partition.

  To conclude, we just have to show that $e(x,y)$ is at most $e(X)$. Assume for contradiction that $e(x,y) \geq a$. We consider the partition into $X_2 := \{x,y\}$ and $Y_2 := V \setminus \{x,y\}$. Observe that the minimum underlying degree in Y_2 is at least two, since G being triangle-free, a vertex of Y_2 can only be joined to at most one vertex of X_2. Thus $e(Y_2) \geq b-2$. By maximality of a, (X_2,Y_2) is not an $(a+1)$-partition, thus $e(X_2,Y_2) > (2m-n)/3+1$, that is $d(x)+d(y)-2e(x,y) > (2m-n)/3+1$.

  Consider now any partition (X_3,Y_3) such that $(x \cup N(x)) \setminus y \subseteq X_3$ and $(y \cup N(y)) \setminus x \subseteq Y_3$. We have $e(X_3) \geq d(x)-e(x,y) \geq n/2$. Similarly $e(Y_3) \geq n/2$. So, if this partition is not good, we must have $e(X_3,Y_3) > (2m-n)/3+1$. Thus $m-(d(x)+d(y)-2e(x,y)) > (2m-n)/3+1$. This gives $m > 2(2m-n)/3+2$, and finally $m < 2n-6$ which is impossible since the minimum degree in G is at least four.

■

5 Conclusion

Note that the only case where $\mathrm{bw}(G) > \mathrm{bw}(M_G)$ is when G has a bridge e and that $\mathrm{bw}(M_G) = 1$, that is when G is a tree that is not a star.

A consequence of our result is a new proof of the fact that the branch-width of a connected planar graph that is not a tree and the branch-width of its dual are the same, for previous proofs see [4] and [1]. Indeed, if r_M is the rank function of a matroid, the rank function r_{M^*} of the dual matroid is such that $r_{M^*}(U) = |U| + r_M(E \setminus U) - r_M(E)$ which implies that $w_M(E_1, E_2) = w_{M^*}(E_1, E_2)$ and that $\mathrm{bw}(M) = \mathrm{bw}(M^*)$. The result follows from the fact that if G is a planar graph and M_G its graphic matroid, the dual matroid of M_G is M_{G^*}.

Note that this dual property also holds for stars and thus the only planar graphs G such that $\mathrm{bw}(G) = \mathrm{bw}(G^*)$ are exactly the planar graphs such that $\mathrm{bw}(G) = \mathrm{bw}(M_G)$. We feels that the natural definition for the branch-width of graphs is the matroidal one.

An independent proof of the equality of branchwidth of cycle matroids and graphs was also given by Hicks and McMurray [3]. Their method is based on matroid tangles and is slightly more involved than ours.

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